CONSTITUTIVE RELATIONS FOR GENERAL FLUID FLOW

Mechanics is a part of natural philosophy studied in depth and in repetition, next only to geometry and in the last four decades, there has been an extraordinary resonance in the literature on rational mechanics, studied as a (mathematical) discipline in its own right. This resurgence has signalled and led the rediscovery of natural philosophy as a whole and it all began in the forties of this century. Daily experience points out that the primitive elements of mechanics are the three concepts : viz., (i) a body (B) - a three-dimensional smooth manifold the elements of which are particles $(\bar{X})$; (ii) Euclidean space-time - consisting of points $(\bar{x})$ endowed with the Euclidean metric structure that is independent of the time (t); (iii) a system of forces.

The motion of a body is the specification of the places taken by the particles $(\bar{X})$ as time goes on or the time sequence of configurations, i.e. the spacial regions occupied by the body. $\bar{x} = \bar{\chi}(\bar{X}, t)$. The forces in continuum mechanics are contact forces which express the reaction of material upon material in a given configuration of a body. The stress vector $\bar{t}_{(n)}$ acting upon a boundary with outer normal $(\bar{n})$ is related to the stress tensor (T) in the form $\bar{t}_{(n)} = T\,\bar{n}$, the latter being an inner product of the tensor T and the unit normal vector $\bar{n}$. The study of mechanics consists in the effort to connect the above three elements: body, motion and force so as to provide the right model(s) for the behaviour of materials in nature. This effort comprises several steps of which the two most prominent are the

conservation laws of force and torque and these are postulates valid for all bodies. When the fields (scalars, vectors and tensors) that enter the discussion are sufficiently smooth, the above principles of linear momentum and the moment of momentum take the form (Cauchy, 1823)

$$\text{div } T + \rho b = \rho \ddot{x}, \quad T = T' \qquad (1)$$

These are valid for all bodies, independent of the material properties and ρ, $\bar{b}$, $\ddot{x}$, $'$ denote the mass density, body force per unit mass, acceleration and transposition of the tensor/(matrix). Two bodies of different material differ in their behaviour even though the same external forces $\bar{b}$ are applied on them and the inference therefore, is that the `internal forces' differ in the two cases. If $\bar{b}$ and $\ddot{x}$ are specified, Cauchy's law leads to the determination of div T and not of T. Thus the above general principles of linear momentum and the moment of momentum, are not adequate to determine the forces acting within a body which is subjected to a given motion.

Constitutive Equations are relations that determine the stress in various materials from sufficient data of motion. What we see to be reasonably valid for one or another material serves as guideline in the choice of a particular class of constitutive equations. The (mathematical) principles to be ensured for the validity of any constitutive equations are (i) Principle of Determinism, (ii) Principle of Local Action and (iii) Principle of Material Frame - Indifference (Material Objectivity).

The stress in a body is determined by the history of the

motion of the body (B).

$$T(\bar{X}, t) = F\{\bar{\chi}; \bar{X}, t\} \qquad (2)$$

and F in (2) is a functional of the motion $\bar{\chi}$ of the body (B), i.e. the stress T is determined by the motion $\bar{\chi}$ in its entirely, i.e. $\bar{\chi}(\bar{X}, \tau)$ with $-\infty < \tau \leq t$, t being the present time. In alternative terms, knowledge of the motion of all particles of the body at the present time and all past times suffices to determine the present stress at the particle $\bar{X}$. Past experience is thus relevant for the determination of the stress. Thus there is no reason to believe that the determination of the stress rests on reversible processes. The constitutive functional F in (2) defines a particular material and expresses all the information we have about the mechanical properties of the particles making up the body.

Stress forces $\bar{t}_{(n)}$ are contact forces and are to be determined by contact actions only, even though, it is possible that such contact actions belong to the (remote) past on the time scale. The functional F in (2) must reflect this trend in material behaviour. If two motions, $\bar{\chi}_1$, $\bar{\chi}_2$ differ only outside the small neighbourhood $N(\bar{X})$ of $\bar{X}$ for all times $\tau \leq t$, the present stress at X arising from $\bar{\chi}_1$ and $\bar{\chi}_2$ are to be the same. i.e. if $\bar{\chi}_1(z, \tau) = \bar{\chi}_2(z, \tau)$ for all $\tau \leq t$ and for $z \in N(X)$,

$$F(\bar{\chi}_1; X, t) = F(\bar{\chi}_2; X, t) \qquad (3)$$

If two observes O_1 and O_2 consider the same motion in a given body, they find the same state of stress. If $\bar{\chi}_1(X, t_1)$ is the motion observed by O_1 and $\bar{\chi}_2(\bar{X}, t_2)$ is the motion observed by

0_2, in the realm of classical mechanics, we have

$$\bar{\chi}_2 = Q\,\bar{\chi}_1 + \bar{c},\quad t_2 = t_1 - a \tag{4}$$

where Q is a time-dependent orthogonal matrix, $\bar{c}$ is a time-dependent vector and a is a constant. The principle of determinism shows that

$$T_2(\bar{X}, t_2) = F\{\bar{\chi}_2;\ X,\ t_2\},\quad T_1 = F\{\bar{\chi}_1;\ X,\ t_1\} \tag{5}$$

where F is the constitutive functional, stipulated once and for all in the definition of the particular material. T_1 and T_2 must yield the same stress vectors at every point $\bar{(X)}$ and for every surface element $(\bar{n})$ - (principle of material - frame indifference); i.e.,

$$T_2 = Q\,T_1\,Q' \text{ and } F\{Q\,\bar{\chi}_1 + \bar{c}\ ;\ X, t_1 - a\} = QF\{\bar{\chi}; X, t_1\}Q' \tag{6}$$

must hold identically in Q, $\bar{\chi}$, $\bar{c}$, a.

A material in which the response for <u>all</u> deformation processes is known by knowing its response for all <u>homogeneous</u> deformation processes, is called a <u>simple</u> material.

Body B has its present configuration $\bar{x}$ and $\bar{X} = \bar{x}(X)$; $\bar{X}$ is the material coordinate and $\bar{x}$ is the (spacial) configurational coordinate.

$$\bar{x} = \bar{\chi}(\bar{X},\ t) = \bar{\chi}\{x^{-1}(\bar{x}),\ t\} = \bar{\chi}_x(\bar{X}, t)$$

$$\bar{F}(\bar{X}, \tau) = \text{grad}\ \bar{\chi} = \frac{\partial}{\partial X}\,\bar{\chi}\{x^{-1}(\bar{x}),\ t\} = \frac{\partial}{\partial X}\,\bar{\chi}_x(\bar{X}, t) \tag{7}$$

gives the gradient of the deformation from x to $\bar{x}$. A deformation history (i.e. the totality of the values of $\bar{F}(\bar{X}, \tau)$ for all times τ upto the present time t) is called homogeneous if $\bar{F}$ is constant in space, at each time. The stress in a simple material is known if we know the history of the gradient $\bar{F}(\tau)$. Thus the constitutive equation of a simple material has the form

$$T = \underset{\tau=-\infty}{\overset{\tau=t}{G}} \{ F(\tau) \} \tag{8}$$

where the constitutive functional G may depend on $\bar{x}$, $\bar{X}$ and t as parameters. The principle of material-frame indifference demands that

$$G(Q(\tau))F(\tau)) = Q(t)G\ (F(\tau))\ (Q(t))' \tag{9}$$

identically in Q and F. The deformation $F(\tau) = R(\tau)U(\tau)$, where R is a rotation and U is a pure deformation, i.e. any deformation $F(\tau)$ can be split as a compound of a pure deformation $U(\tau)$ followed by a (rigid) rotation $R(\tau)$. Hence, we have

$$G\{Q(\tau)R(\tau)U(\tau)\} = Q(t)\ G\{R(\tau)U(\tau)\}(Q(t))'$$

Choice of $Q(\tau) = (R(\tau))'$ shows that

$$G\{U(\tau)\} = (R(t))'\ G\{F(\tau)\}\ R(t) \tag{10}$$

The constitutive equation of a simple material has, therefore the form:

$$T = R(t)\ \left(\underset{\tau=-\infty}{\overset{\tau=t}{G}} \{U(\tau)\} \right)\ (R(t))' \tag{11}$$

The stress tensor may depend in any way on whatever pure

deformation the particle has experienced and past rotations have no effect on the present stress. The material point X of the body B is simple if the outcomes of all experiments at X are determined by the outcomes of all experiments on homogeneous deformations of parts of B near X. Laboratory programmes of experimental mechanics swear by homogeneous deformations and results of more complicated deformations are commonly explained in terms of them. Thus, consciously or otherwise, experimenters presume that a laboratory material may be modelled sufficiently well in the mathematical theory by some simple material. The theory of simple materials includes all the common purely mechanical theories of continua studied in mechanics. Appropriate generalization is possible to take account of heating and change of temperature.

An elastic material is a special type of simple material in which the mapping G in (11) is a function g of the present local deformation $\bar{F}(\bar{X}, t)$. By polar decomposition the local deformation $F = R\,U$ and hence the result: The constitutive relation of an elastic material has the form

$$T = R\,g\,(U, X)\,R' \qquad (12)$$

in which $g(U, X)$ maps positive definite symmetric tensors onto symmetric tensors (Cellerier, Richter). The special choice of $g(U) = g(\det U)\,I$ leads to the relation

$$T = -p(\rho)\,I \qquad (13)$$

p being the pressure field, which is determined by the density

field. The material defined by (13) is an elastic fluid or Eulerian fluid. The velocity field of a fluid is called a flow and the popular meaning of this term reconciles with the above definition by noting that a fluid cannot sustain shear stress when at rest in any configuration. This property is common to all simple fluids, but cannot adequately define a simple fluid. An Eulerian fluid cannot sustain shear stress, be it in a state of rest or in a state of motion.

A class of fluids more general than Eulerian, is defined by the constitutive relation

$$T = h\ (G,\ \rho,\ \bar{x},\ x,\ t) \qquad (14)$$

where h is a functional of the arguments : $G = \text{grad}\ \bar{\dot{x}}$ = the velocity gradient, ρ = density, $\dot{\bar{x}}$ = velocity. The symmetric part of G is $D = 1/2\ (G + G')$ is the rate of change of the stretch (at X) and $W = 1/2\ (G-G')$ = the antisymmetric part of G = the spin tensor. Within a convention of sign, W may be represented by the axial vector (pseudo - vector) curl x. The tensor D is the rate of deformation. The scalar E called the expansion = div $\dot{x}$ = tr G = tr D = $-\dot{\rho}/\rho$. The principle of material - indifference compels that

$$h(G, \rho, \dot{\bar{x}}, \bar{x}, t) = Q' h(D^* + W^*, \rho^*, \dot{\bar{x}}^*, \bar{x}^*, t^*)\ Q$$

$$= Q' h(QDQ' + QWQ' + A, \ , Q\dot{\bar{x}} + \bar{x}_o^* + A(x^* - x_o^*),\ x_o^{*\bar{}} + Q(x - x_o)^{\bar{}}, t+a)Q$$

Keeping the arguments $G, \rho, \dot{x}, x, t$ fixed and choosing $\{Q(t) = I,$

$$\bar{A} = \dot{Q}(t) = -W,\ x_o^* = -x^{\bar{}} - A(x^{\bar{}} - x_o^*(t)),\ x_o^{*\bar{}}(t) = x_o - (x - x_o)^{\bar{}},$$

$a = -t$} shows that

$$h(g, \rho, \bar{x}, \bar{x}, t) = h(D, \rho, 0, \bar{x}_o, 0) \qquad (16)$$

x_o being any fixed point. Thus h is a function of D and only.

$$h(G, \rho, \bar{x}, \bar{x}, t) = h(D, \rho) \qquad (17)$$

The spin W, the velocity $\dot{\bar{x}}$ may be transformed away by a suitable change of frame; any place ($\bar{x}$) and time (t) may similarly be - transformed into any other place and time. The four arguments W, $\dot{\bar{x}}$, $\bar{x}$, t cannot explicitly enter in a constitutive relation. Further, we must have the following relation as an identity for all symmetric tensors D and all orthogonal tensors Q.

$$h(QDQ', \rho) = Q\, h(D, \rho)\, Q' \qquad (18)$$

<u>Noll's theorem</u>. The relation (14) conforms to the principle of material frame - indifference if and only if the functional h reduces to a function of D and ρ only and (18) is satisfied as an identity in Q and D.

$h(D, \rho) = T$ may be interpreted as a mapping of the (symmetric) tensor D onto the tensor T which conforms to invariance under any orthogonal transformation (Q). Noll's theorem states that all those materials whose constitutive relations are included in (14) are <u>isotropic materials</u>. If h is restricted to be an affine function (i.e., a linear function) of the argument D, we have a <u>linearly viscous fluid</u>.

<u>(Cauchy)</u> : In order that a function h mapping symmetric tensors onto symmetric tensors is isotropic and also affine, it is both

necessary and sufficient that h has the forms

$$h(A) = (\alpha + \beta \text{ tr } A)\, I + \gamma\, A \qquad (19)$$

in which α, β, γ are constants.

Combining the theorems of Cauchy and Noll we have theorem (Stokes) : The constitutive relation of a linearly viscous fluid is of the form

$$T = (-p + \lambda \text{tr } D)\, I + 2\mu\, D \qquad (20)$$

where p, λ, μ are functions of the scalar ρ . Every such relation defines a linearly viscous fluid.

The study of fluid flow based on the above constitutive law (20) is the Navier-Stokes theory of fluid flow. Navier, Cauchy, St. Venant and Stokes have each, derived one or another major special case of the above law under diverse hypotheses. The coefficients λ and μ are the viscosities of the fluid. In rigid motions D = 0 and the Navier-Stokes theory reduces to Eulerian hydrodynamics of inviscid or perfect fluids. The fluid it defines exhibits the phenomenon of `flow', viz., in a state of rest it can sustain only hydrostatic stress. If $\lambda = 0$, $\mu = 0$, the linearly viscous fluid reduces to an elastic fluid.

Consider places that are values of the motion of X at two times t and τ.

$$\bar{\xi} = \chi(\bar{X}, \tau), \quad \bar{x} = \chi(\bar{X}, t) \qquad (21)$$

Then $$\bar{\xi} = \chi(\chi^{-1}(\bar{x}, t), \tau) = \chi_t(x, \tau) \qquad (22)$$

The local relative deformation is

$$F_t = F_t(\chi) = \text{grad}\, \chi_t \qquad (23)$$

(c.f. eqn. (7)). Then $F_t(t) = I$ and at X, $F(\tau) = F_t(\tau)\, F(t)\, F = R\, U = V\, R$ where R is an orthogonal tensor and U, V are both positive symmetric tensors, called respectively the right and left stretch tensors.

$$F = R\, U = V\, R,\ V = RUR',\ U = R'F$$

The tensors $C = U^2 = U'U = F'\, F$ and $B = V'\, V = RCR'$ are the right and left Cauchy - Green tensors.

If the class of displacements is limited at interior points of the body, the material is said to be subject to internal constraints. A simple constraint is expressed by the requirement that the local deformation F satisfies an equation of the form $\mathcal{N}(F) = 0$, where $\mathcal{N}$ is a frame-indifferent scalar function. Equivalently, we have $\lambda(C) = 0$ where λ is a scalar function.

A material is said to be incompressible if it can experience only isochoric motions. The constraint function is

$$\lambda(C) = \det C - 1.$$

(A motion of B is called isochoric if the volume $V(\chi(P,t))$ of the shape of each part P of the body B remains constant in time). The stress in this case is

$$T = -\, p\, I + G\,(F^t) \qquad (24)$$

where p is an arbitrary scalar. In an incompressible material

the stress is determined from the motion only to within an arbitrary hydrostatic pressure (Poincaré).

Inextensibility of a material in a particular direction is another form of constraint. If e_k is a unit vector in this direction the homogeneous transplacement with gradient F carries e_k into Fe_k and the appropriate constraint function is

$$\lambda(C) = |Fe_k|^2 - 1 = e_k . C e_k - 1 \qquad (25)$$

The stress T is determinate only to within a uniaxial tension in that direction.

The material is rigid if it is inextensible in every direction. The stress is then determinable only to within an arbitrary tension in every direction. The stress is therefore unaffected by the motion. The rigid motion of anybody is determined without knowledge of whatever may be the stress.

The constitutive relation of an incompressible elastic material has the form

$$T = - p I + R g (U) R', \quad \det U = 1 \qquad (26)$$

For an incompressible elastic fluid, the constitutive relation is $T = - p I$ and for an incompressible linearly viscous fluid, it is

$$T = -p I + 2 \mu D, \quad \text{tr } D = 0 \qquad (27)$$

In each of these three cases, the hydrostatic pressure p is indeterminate and may be assigned independently of the history of the motion.

Material Isomorphism : If X_1, X_2 are two distinct points of body B, when are they of the same material? If it is possible to bring the portions P_1, P_2 of B near the points X_1, X_2 into reference shapes $k_1(P)$, $k_2(P)$ respectively such that any subsequent deformation history gives rise to exactly the same stress at the places $\bar{X}_1$, $\bar{X}_2$ occupied by the points (X_1, X_2) we may say that the points X_1, X_2 are of the same material. In such a case no experimental programme measuring the stress as determined by the deformation) can detect whether we started with the part P_1 containing $\bar{X}_1$ in $\bar{k}_1(B)$ on the part P_2 containing $\bar{X}_2$ in $\bar{k}_2(B)$, as long as $\bar{X}_1 = \bar{k}_1(X_1)$, $\bar{X}_2 = \bar{k}_2(X_2)$. We must have of course, $\rho_{\bar{k}_1} = \rho_{\bar{k}_2}$; the densities are to be uniform near X_1 anx X_2.

Let G_k be the response of a simple material with respect to the reference placement $\bar{k}$. The points X_1, X_2 of B are materially isomorphic if these are reference placements $\bar{k}_1$, $\bar{k}_2$ such that $\rho_{\bar{k}_1} = \rho_{\bar{k}_2}$ = constant near $\bar{X}_1$ and $\bar{X}_2$ and for every history F^t in a suitable class

$$G_{\bar{k}_1}(F^t, \bar{X}_1) = G_{\bar{k}_2}(F^t, \bar{X}_2) \qquad \text{(Noll)} \qquad (28)$$

If the isomorphism of all the points of a uniform body may be demonstrated by use of a single reference placement $\bar{k}$, the body is homogeneous. All homogeneous bodies are uniform, but the converse is not true. Uniform but non-homogeneous bodies correspond to bodies with "defects" and "dislocations".

Isotropic Materials. Isotropy is an important example of

material symmetry. A homogeneous body is isotropic if it can be brought into a shape, no rotations of which can be detected by experiment. The constitutive equation for an isotropic material is of the form (Noll)

$$T = K\ (C_t^t\ ;\ B) \qquad (29)$$

where $C = F^t\ F$ and $B = F\ F^T$ are the right and left Cauchy-Green tensors.

A fluid is a non-solid egalitarian material. It is necessarily isotropic. It is a material having no preferred configuration. It is unable to support shear stress when it is in a state of equilibrium. Within the theory of simple materials, we have

$$T = -p(\rho)\ I + \mathcal{S}(F^t) \qquad (30)$$

Navier - Stokes equations : They embody "the simplest hypothesis we can frame" and "this hypothesis has been put to a very severe test in the experiments of Poiseuille and others [so that] we can hardly hesitate to accept the equations in question as a complete statement of the laws of viscosity" (Quote : Sir Horace Lamb :Hydrodynamics, Camb. Univ. Press (1895), § 326). The particular consequence of the Navier - Stokes theory which the experiments of Poiseuille and others tested is the law of the fourth power for the discharge D of an incompressible fluid of shear viscosity μ from a circular pipe of radius R through which it is driven by a constant pressure gradient = A.

$$D = \frac{\pi A R^4}{8\mu}$$

This formula is deduced using the constitutive relation $T = -pI + \mu A_1$, where A_1 = grad $\dot{x}$ + (grad $\dot{x}$)' = twice the stretching tensor = 2D. Assumptions that μ and ρ are constant imply that a single experiment measuring the discharge from one particular pipe subject to one particular pressure gradient suffices to determine μ and thereby exhausts all the physical properties the fluid has. For, with the determination of μ, the differential equations of motion with appropriate boundary conditions can be integrated under various other circumstances. The motion along the length of the pipe is determined by solving the standard Dirichlet problem for the speed W which satisfies a Poisson equation.

In 1943 (NATURE, Vol. 152, p. 663) A.C. Merrington reported the experimental observation of the tendency for a certain fluid stream to swell at the exit section of a viscometer and the swelling is emphasized by high flux and small pipe radius. Certain experiments of Weissenberg (NATURE, Vol. 159, p. 310, (1947)) revealed the so-called climbing effect of some fluids when set in Couette motion by the rotation of a cylinder inside a container filled with the fluids. The Navier - Stokes equations do not support either of these observations.

Suppose instead that the fluid model is non-Newtonian with the constitutive relation

$$T = \alpha I + \beta D + \gamma D^2 \qquad (31)$$

D being the rate of deformation tensor and α, β, γ are scalars. For incompressible fluids α is identified as the negative (= -p) of the dynamic pressure and β, γ are presumed to be functions of the invariants II, III of the tensor D and of the thermodynamic state of the fluid. T = fluid temperature

$$\beta = \beta\ (II,\ III,\ T),\quad \gamma = \gamma\ (II,\ III,\ T) \tag{32}$$

The familiar Cauchy law (1) viz., $\rho\ dq/dt = \rho\ b + divT$ is then employed to analyze the flow with the fluid defined by the constitutive relation (31).

<u>Poiseuille Flow</u> Steady laminar flow down a straight pipe of diameter = 2a. (r, Θ, z) $q_r = 0$, $q_\Theta = 0$, $q_z = W = W/r)$, $0 \le r \le a$

$$D = \frac{1}{2}\begin{pmatrix} 0 & 0 & W' \\ 0 & 0 & 0 \\ W' & 0 & 0 \end{pmatrix},\quad T = -\beta I + \frac{1}{2}\beta W'\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} +$$

$$\frac{1}{4}\ W'^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{34}$$

$\beta = \beta\ (\frac{1}{2}\ W'^2\ ,\ 0\ ;\ T) = \beta(r,T)$, $\gamma = \gamma\ (r,T)$. The acceleration is zero and the equations of motion can be integrated with $\bar{b}$ (body force) $= -g\bar{k}$. We have

$$p = (C-\rho g)z + f(r),\quad \beta W'(r) = Cr,$$

$$f(r) = \frac{C^2}{4}\left(\frac{\gamma r^2}{\beta^2} + \int \frac{\gamma r^2}{\beta^2}\,dr\right),\ W(a) = 0 \tag{35}$$

The stress components of interest are

$$t_{11} = t_{33} = -p + \frac{1}{4}\gamma W'^2 = (\rho g - C)z - \frac{C^2}{4}\int \frac{\gamma r}{\beta^2}\,dr$$

$$t_{13} = \frac{1}{2} Cr$$

and the total mass flux is

$$M = -\int\int \rho\, W r dr d\Theta = \pi C \rho \int_0^a (r^3/\beta) dr \qquad (36)$$

The normal stress t_{11} ($= \sigma_{rr}$) on the pipe wall varies linearly along the wall and also depends on the non-Newtonian viscosity γ (the so-called cross viscosity).

If β and γ are constants, we have the classical results

$$W = \frac{C}{2\beta}(r^2 - a^2), \quad M = \frac{\pi \rho C}{4\beta} a^4, \quad \beta = 2\mu \qquad (37)$$

and the absence of γ in these formulae shows that the standard viscometric measurements will not indicate the presence or absence of the non-linear viscosity of the fluid. This parameter (γ) however appears in the non-uniform distribution of pressure across a section of the pipe. Keeping the constancy of the coefficients β, γ we see that

$$t_{11} = t_{33} = (\rho g - C)\, z - 2\gamma \left(\frac{C}{8\mu}\right)^2 r^2 - \text{const.} \qquad (38)$$

and at the exit section (chosen as z = 0)

$$\pi a^2 p_o = -2\pi \int_o^a (t_{33}) r dr = \pi \gamma \left(\frac{C}{8\mu}\right)^2 a^4 + \text{constant} \tag{39}$$

$$t_{11} = t_{33} = (\rho g - C)z + \left(\frac{C}{8\mu}\right)^2 (a^2 - 2b^2) - p_o \tag{40}$$

If P = force per unit area which the fluid exerts on the pipe walls,

$$\text{(i.e. } P = -t_{11} \text{(at } r = a)\text{)}, \quad P = p_o - 2\rho z + \left(\frac{\Gamma}{a}\right)^2 \left(\frac{8\mu}{\Gamma} z + \gamma\right) \tag{41}$$

with $\Gamma = M/\pi\rho a^2$ = volume of flow per second per unit cross sectional area. Formula (41) gives a theoretical explanation of the bulging effect of Merrington at the exit (z = 0) of the viscometer with

$$(P - p_o)\ (z = 0) = \left(\frac{\Gamma}{a}\right)^2 \gamma \tag{42}$$

A positive value of explains the occurrence of the excess pressure. If the coefficients, β, are not assumed to be constant, we have

$$t_{11} = t_{33} = (\rho g - z) = -\left(\frac{C}{2}\right)^2 \int_a^r \frac{\gamma}{\beta^2} dr - \text{const.} \tag{43}$$

and at the exit section

$$\pi a^2 p_o = -2\pi \int_o^a (t_{33}) r dr = -\pi \left(\frac{C}{2}\right)^2 \int_o^a \frac{\gamma}{\beta^2} r^3 dr + (\pi a^2)\text{constant} \tag{44}$$

$$t_{11} = t_{33} = (\rho g - C)z - \frac{C^2}{4}\left(\int_a^r \frac{r\gamma}{\beta^2}\,dr + \frac{1}{a^2}\int_o^a \frac{a\gamma r^3}{\beta^2}\,dr \right) - p_o \qquad (45)$$

and $P - p_o = \left(\frac{C}{2a}\right)^2 \int_o^a \frac{a\gamma r^3}{\beta^2}\,dr$ at the exit section.

The swelling of the fluid stream is clear as long as $\gamma > 0$, irrespective of whether β and γ are constants or not.

<u>Couette flow</u> (r, Θ, z), $q_r = q_z = 0$, $q_\Theta = rw(r)$ (46)

$$D = \frac{1}{2}\begin{pmatrix} 0 & rw' & 0 \\ rw' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D^2 = \frac{1}{4}\begin{pmatrix} r^2w'^2 & 0 & 0 \\ 0 & r^2w'^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (47)$$

$$\beta = \beta\left(-\frac{r^2w'^2}{4}, 0 ; T\right) = \beta(r,T), \quad \gamma = \gamma(r,T) \qquad (48)$$

Assuming that β and γ do not explicitly involve the temperature T (or if T depends only on r),

$$p = -\rho gz + f(r), \quad r^3\beta w' = C, \quad f(r) = \left(\frac{C}{2\beta}\right)^2 \frac{\gamma}{r^4} + \rho \int rw^2 dr \qquad (49)$$

$$v = rw = Ar^{-1} + Br, \quad A = -C/2\beta$$

Imagine the flow between concentric cylinders in rotation with the upper fluid surface being left open to the atmosphere. What is the shape of this free surface?

$$t_{33} = -p = \rho gz - A^2 r^{-4} - \rho \int (A + Br^2)^2 r^{-3}\,dr \qquad (50)$$

The shear stress t_{12} does not vanish. If the free surface is nearly horizontal and remains so, its locus is given by t_{33} = constant and its slope satisfies the relation

$$\frac{dz}{dr} = \frac{A^2}{gr^5}\left\{\left(\left(1 + \frac{B}{A}r^2\right)r\right)^2 - \frac{4\gamma}{\rho}\right\} \qquad (51)$$

If the inner cylinder ($r = r_1$) is stationary $B/A = -1/r_1^2$ and equals $-1/r_2^2$ if the outer cylinder is at rest. When = 0, the slope varies from 0 at the inner cylinder to a +ve value at the outer cylinder and in the other case, it varies from a +ve value at the inner cylinder to zero at the outer cylinder.

1. When $\gamma > 0$, the fluid tends to a climb the inner cylinder and also climbs the outer cylinder unless γ is very large. (this is when the inner cylinder is stationary).

2. When the outer cylinder is stationary, the fluid tends to fall slightly at the outer cylinder. At the inner cylinder the fluid tends to rise or fall according as

$4\gamma - \rho\left\{\left(1-\left(\frac{r_1}{r_2}\right)^2\right)r_1\right\}^2$ is greater than or less than zero.

If $r_1 \ll r_2$, the fluid tends to climb the inner cylinder if $r_1 < 2\sqrt{\frac{\gamma}{\rho}}$

and to fall away from the inner cylinder otherwise.

These conclusions are in general agreement with the experimental results of Weissenberg.

An upper limit to the Navier - Stokes Theory

According to the kinetic theory of monatonic, moderately rarefied gases, the Navier - Stokes constitutive equation is just an approximation. The arguments that support the above rest on formal processes of approximation to solutions of the Maxwell - Boltzman equation, which belong to a hypothecated special class. It is therefore necessary to have a precise set of assumptions when the kinetic theory reduces it to the Navier - Stokes theory and emphasize that these assumptions must hold in order that the Navier - Stokes theory may follow.

Modern continuum mechanics compels us to expect that departures from classical predictions manifest themselves in normal - stress effects. The definition of the pressure tensor in the kinetic theory forces the normal pressure on every element of area in every element of area in every flow to be positive, unless the mass density vanishes but the Navier - Stokes theory makes a tension result on elements which suffer sufficiently violent distortion. The two theories do not reconcile with each other when the normal-stress effects in the two theories are contrary.

The linear constitutive relation between the stress T and the rate of deformation tensor D has the form

$$T = -p\, I + \lambda(\operatorname{div} \bar{q})I + 2\mu\, D \qquad (52)$$

where $\bar{q}$ is the fluid velocity vector and $2D = \{(\operatorname{grad} \bar{q}) + (\operatorname{grad} \bar{q})^T\}$.

This general linear law is never recoverable from the kinetic theory unless the Stokes' condition is adopted, viz., $3 + 2\mu = 0$. Then the pressure tensor P is related to the stretching tensor D in the form

$$P = \left(p + \frac{2\mu}{3} \text{ tr } D\right) I - 2\mu D \tag{53}$$

The pressure p and the shear viscosity μ are positive functions of density and temperature and the normal pressure on an element having $\bar{n}$ as the unit normal vector, is

$$\begin{aligned} \bar{n} P \bar{n} &= \bar{n} \left\{ \left(p + \frac{2\mu}{3} \text{ tr } D\right) I - 2\mu D \right\} \bar{n} \\ &= p + 2\mu \bar{n} \left(\frac{1}{3} (\text{tr } D) I - D \right) \bar{n} \end{aligned} \tag{54}$$

and this quantity is positive if and only if

$$2\bar{n} (D_o) n < p/\mu \tag{55}$$

$$D_o = \text{deviator distortion tensor} = D - \frac{1}{3} (\text{tr } D) I$$

Since (55) must hold for all $\bar{n}$, it follows that the maximum of the expression $2 \bar{n} (D_o) n$ must be less than p/μ. If D has the eigenvalues d_1, d_2, d_3 ordered such that $d_1 \geq d_2 \geq d_3$ (d_1, d_2, d_3 are the principal rates of deformation), the greatest eigenvalue of D_o is $1/3 \{(d_1-d_3)+(d_1-d_2)\}$ and the inequality (55) holds if and only if

$$Te = \frac{2}{3} (2d_1 - d_2 - d_3) \frac{\mu}{p} < 1 \qquad (56)$$

Te is the tension number and no element of area is suffering positive tension if and only if Te < 1. The Navier - Stokes constitutive equation cannot be consistent with the Maxwellian kinetic theory unless Te < 1. The tension number Te is defined in terms of gross quantities and therefore can be used in continuum theories as well as kinetic theories.

To estimate the error in cutting off the constitutive equations after the linear terms, we may introduce the truncation number

$$Tr = \{2 (d_1^2 + d_2^2 + d_3^2)\}^{1/2} (\mu/p) \qquad (57)$$

The two numbers Te and Tr are such that

$$(Te) \leq \frac{2}{\sqrt{3}} (Tr) \qquad (58)$$

and the two numbers are equal for a general spatial flow if and only if $d_2 = d_3 = -1/2\ d_1$.

In isochoric motions $d_1 + d_2 + d_3 = tr\ D = 0$. For plane isochoric flow, $d_2 = 0$ and $d_3 = -d_1 \leq 0$ and then Te = Tr. A simple shearing motion given by $\underline{q} = (u, v, w) = (ky, 0, 0)$ gives $d_1 = 1/2\ k$ and k is the amount of shearing. Then

$$Te = Tr = \frac{\mu k}{p} = k \left(\frac{\mu}{p}\right) = k\ \tau \qquad (59)$$

and $\tau = \mu/p$ = the relaxation time. The limit Te < 1 shows (asserts) that the Navier - Stokes constitutive equation fails before the shearing (= k) becomes as great as the relaxation rate = $(1/\tau = p/\mu)$.

For a Maxwellian gas $\tau = \tau_o\ (p_o/p)$. If we take $\tau_o = \mu/p$ for air at standard conditions, the value of k which brings Te to its limiting value is

$$k_{limit} = \frac{1}{2}\left(\frac{\rho}{\rho_o}\right) \times 10^{10}\ sec^{1}.$$

In a vacuum of 10^{-6} atm, $k_{limit} = 5000\ sec^{-1}$.

No matter what the molecular model is, the condition (56) cannot possibly hold as a general consequence of the kinetic theory for all flows.

Meaning of order

$$\text{If } f(x) = ax + bx^2 + \ldots \qquad (60)$$

the terms ax, bx^2 are the contributions of the first and second orders, respectively, in x. The idea of 'order' is mathematical, not physical and has nothing to do with the occurring phenomenon; there is thus a need for an exact theory of the phenomenon. The normal stresses needed to maintain a state of simple shear within the elastic range are 'second-order' effects as they are proportional to the square of the amount of shear. However, in the classical theory of finite deformation, every solution is a first order solution only and the occurrence of normal stresses

in shear is a 'first order' effect. Thus the same experimental fact of occurrence of normal stresses is consistent with both the theories that reckon the phenomenon as second-order or first-order effects.

M. Reiner [Am. J. Mathematics (67), 1945, 350-362] was the first to propose a non-linear constitutive relation (for an isotropic fluid) in the form

$$T = -p\, I + \alpha_o\, I + \alpha_1\, D + \alpha_2\, D^2 \qquad (61)$$

with α_o, α_1, α_2 dependent on the principal invariants of the stretching (or rate of deformation matrix) D. While the above relation seems almost an automatic extension of the classical linear law of stress - strain-rate, it nevertheless has proved to be the opener of the modern theory of continuum mechanics. for the simple shear flow with shearing amount equal to k,

$$\bar{q} = (u, v, w) = (ky, 0, 0) \qquad (62)$$

$$D = k \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D^2 = k^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T = (-p + \alpha_o)\, I + k\alpha_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + k^2\, \alpha_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (63)$$

The principal invariants of D are tr D = 0, II = k^2, III = 0 and the coefficients α_o, α_1, α_2 are even functions of the shearing.

$$T_{12} = k\alpha_1 = \text{an odd function of } k$$

$$T_{11} - T_{33} = T_{22} - T_{33} = k^2\alpha_2 = \text{an even function of } k. \tag{64}$$

Departure from the classical linear relation between shear stress and shearing is an effect of third or higher odd order in the shearing; the second-order effect in simple shearing is the occurrence of deviatoric normal pressures on the shearing planes and the normal stress effects (which occur when $\alpha_2 \neq 0$) will be of second or higher even order in k. Further, it is significant that while it is usual to regard a simple shearing flow as a one-dimensional phenomenon the shearing effects and normal-stress effects cover all the three spatial dimensions.

The experimental results presented by K. Weissenberg [Nature, 159, (1947), 310-311] showed that in the flow between rotating cylinders the free surface does not remain plane (it should, according to the theory of linear viscosity when the weight and surface tension of the fluid are negligible) and in many cases the fluid rose on the inner cylinder. The diagrams of Weissenberg were clear proof that the Navier - Stokes equations are not the last word in theoretical fluid mechanics and that shearing is not always a one-dimensional phenomenon.

For an incompressible fluid of Reiner's type (constitutive equation (61); fluids with this constitutive relation, later came to be referred to as Reiner - Rivlin fluids) Rivlin obtained the exact solutions for simple shearing flow in a plane - parallel channel, flow in a circular pipe and flow between rotating cylinders [Proc. Roy. Soc. A 193(1948) 260-281; Proc. Camb. Phil.

Soc. 45, (1949), 88-91].

A major stream in the modern theory of materials was Noll's work in 1955, the first to recognize the principles of invariance and their role in the reduction of constitutive assumptions to manageable form. The main and most commonly accepted requirement is the principle of material indifference (principle of objectivity) viz., that a constitutive equation must have the same form for all observers, i.e. it should be insensitive to rigid motion of the body as a whole. Noll showed that if a fluid is defined as a material in which

$$t = f(\bar{q}, w, D) \tag{65}$$

with q = the velocity, w = spin and D = the rate of deformation tensor, it necessarily reduces to Reiner's forms (61). Stokes had originally assumed that translation and spin have no effect on viscosity and Reiner had assumed the isotropy of the material. Noll's result asserted that a relationship of the type (63) precludes anisotropy and that it is impossible for translation and spin to affect the material response. Constitutive equations of special functional forms specifying how many derivatives are to be present are unmotivated definitions, the defining properties of materials are statements of special kinds of invariance; beyond this feature of invariance, the only additional requirements is a principle of determinism. For a simple material the constitutive equation has the form

$$t(=T) = \underset{s=o}{\overset{\infty}{G}} \; F(t-s) \qquad (66)$$

where G is a tensor-valued functional of its tensor argument and F is the deformation gradient from the reference configuration to that in which the particle finds itself at time t-s. Taking the principle of material indifference into account, we have

$$F(T)\, F' = \underset{s=o}{\overset{\infty}{G}} \; C(t-s) \qquad (67)$$

where $\bar{G}$ is an unrestricted functional and C(t-s) is Green's tensor measuring the finite deformation from the reference configuration to that at time t-s. The stress at a particle is determined by the cumulative deformation history of that particle. The theory is extremely general since it takes cognisance of elasticity, viscosity, long-range memory, short-range memory and stress relaxation in any combination. The form of the functional $\bar{G}$ depends upon the reference configuration. The group of all transformations under which the material properties represented by $\bar{G}$ are invariant is the isotropy group of the material, with respect to the reference configuration and this group is a subgroup of the unimodular group.

A simple material is isotropic if there exists for it a reference configuration with isotropy group containing the proper orthogonal group.

A simple material is a simple fluid if its isotropy group, for every reference configuration, is the full unimodular group.

All simple fluids are therefore isotropic.

Second-order stresses and flows of simple fluids

The steady flows of simple fluids for which exact solutions are known provide an instance to see the second-order effects. From (64) based on Reiner's (or Reiner-Rivlin's, to be more truthful) constitutive equation (61) we see that the normal stress T_{11} and T_{22} are equal in the simple shear flow

$$q = (u, v, w) = (ky, 0, 0) \qquad (68)$$

whatever be the values of the response functions α_o, α_1, α_2 in (61). i.e. T_{11} = the normal stress on the planes perpendicular to the flow direction and T_{22} = the normal stress on the shearing planes are equal. On the basis of arguments that are not well understood, Weissneberg [Nature 159 (1947)] asserted that the normal stresses T_{22} and T_{33} are equal and not $T_{11} = T_{22}$ as noticed above. This means that the natural stresses on all planes parallel to the flow (x - direction in the above) are equal. However, neither of these equalities is experimentally confirmed in the case of fluids exhibiting normal-stress effects. To get over this difficulty, Rivlin and Ericksen proposed [J. Rational Mech. Anal. 4(1955), 323-425] an extended definition of fluids with a constitutive equation in which the stress (T) is influenced by the gradients of acceleration upto an order = n, let us say. The Rivlin-Ericksen tensors of acceleration gradients are defined by the recursive relations:

$$A_{n+1} = \frac{d}{dt} A_n + (\text{grad } \bar{q}) A_n + A_n (\text{grad } \bar{q})^T, \; A_o = I \tag{69}$$

and the Rivlin-Ericksen constitutive relation has the form

$$T + pI = f(A_1, \ldots, A_n) \tag{70}$$

where f is an isotropic function of its arguments. The Reiner-Rivlin relation ($T_{11} = T_{22}$) and the Weissenberg relation ($T_{22} = T_{33}$) are now recoverable as special cases. The relation (70) does not involve any sense of approximation and provides a suitably invariant definition of a fluid for all kinds of motion. Accordingly, (70) defines the Rivlin-Ericksen fluid of complexity (n) and n-th order derivatives of the velocity field are involved in the relation (70) and (n+1)th order derivatives appear in the equations of motion. The Reiner - Rivlin fluid (constitutive equation (61)) is a fluid of complexity 1. Rivlin has found the exact solutions for viscometric flows of fluids defined by (70) above and in all these cases $A_3 = A_4 = \ldots = A_n = 0$, so much so that the constitutive relations for fluids of complexity (n = 2) are defined by the equation

$$\begin{aligned} T = {} & -pI+\alpha_1 A_1+\alpha_2 A_2+\alpha_3 A_1^2+\alpha_4 A_2^2+\alpha_5 (A_1 A_2+A_2 A_1) \\ & \alpha_6 (A_1^2 A_2+A_2 A_1^2)+\alpha_7 (A_1 A_2^2+A_2^2 A_1)+\alpha_8 (A_1^2 A_2^2+A_2^2 A_1^2) \end{aligned} \tag{71}$$

where $\alpha_1, \ldots, \alpha_8$ are polynomial functions in the ten scalar invariances { tr A_1, tr A_1^2, tr A_1^3, tr A_2, tr A_2^2, tr A_2^3, tr $A_1 A_2$, tr A_1^2, A_2, tr $A_1 A^2$, tr $A_1 A_2^2$, tr A_1^2, A_2^2}

For the shear flow (rectilinear flow) $q = (ky, 0, 0)$

(72)

$$A_1 = \begin{pmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_n = 0 \quad (n \geq 2) \tag{73}$$

$$\text{tr } A_1^2 = 2k^2, \ \text{tr } A_2 = 2k^2, \ \text{tr } A_2^2 = 4k^4, \ \text{tr } A_2^3 = 8k^6, \ \text{tr } A_1^2 A_2 = 2k^4, \ \text{tr } A_1^2 A_2^2 = 4k^6 \tag{74}$$

$$t_{11} = -p + \alpha_3 k^2, \ t_{22} = -p + k^2 \{(2\alpha_2+\alpha_3)+4(\alpha_4+\alpha_6)k^2 + 8\alpha_8 k^4\}, \tag{75}$$

$$t_{12} = k \{\alpha_1 + 2\alpha_5 k^2 + 4\alpha_7 k^4\}, \ t_{33} = -p, \ t_{31} = t_{23} = 0$$

In the absence of body forces on the fluid, p = a constant.

descend so as to extrude the contents of the reservoir [illegible]

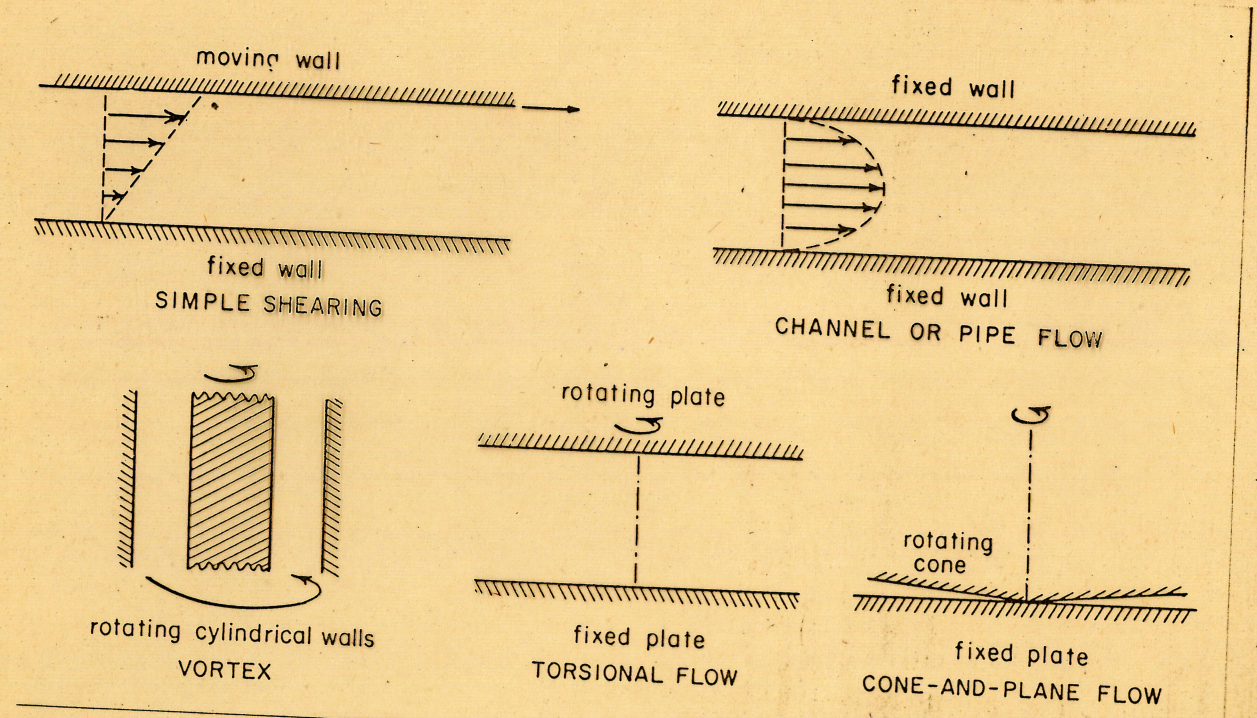

Figure 4 Schematic diagram of some typical viscometric flows.

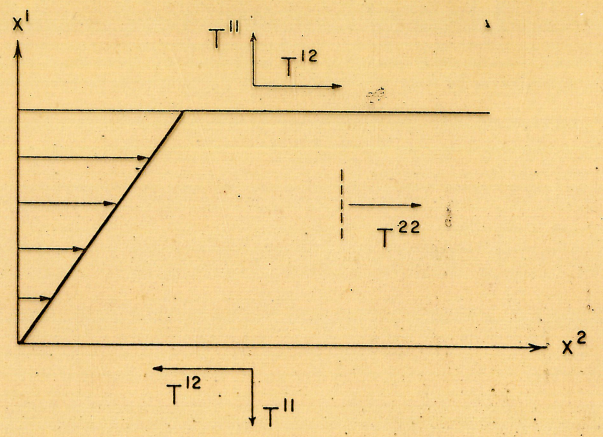

Figure 6 Simple shearing.

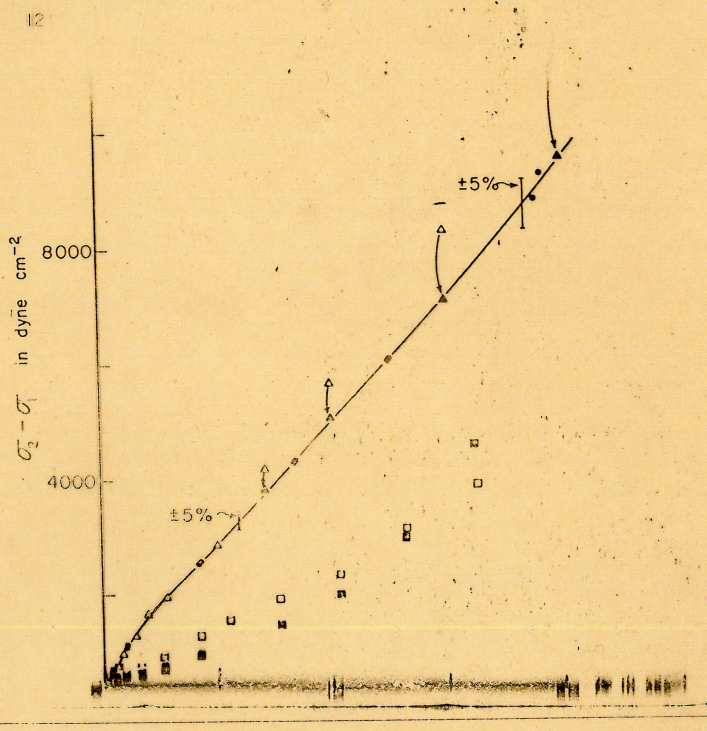

Figure 7 (Courtesy of Professor Pritchard) Measured values of $\sigma_2 - \sigma_1$ as a function of κ in a solution of polyisobutylene of high molecular mass in polyisobutylene of low molecular mass. The triangles, squares, etc., pertain to different laboratories. The curve connects the values Professor Pritchard regards as most reliable.

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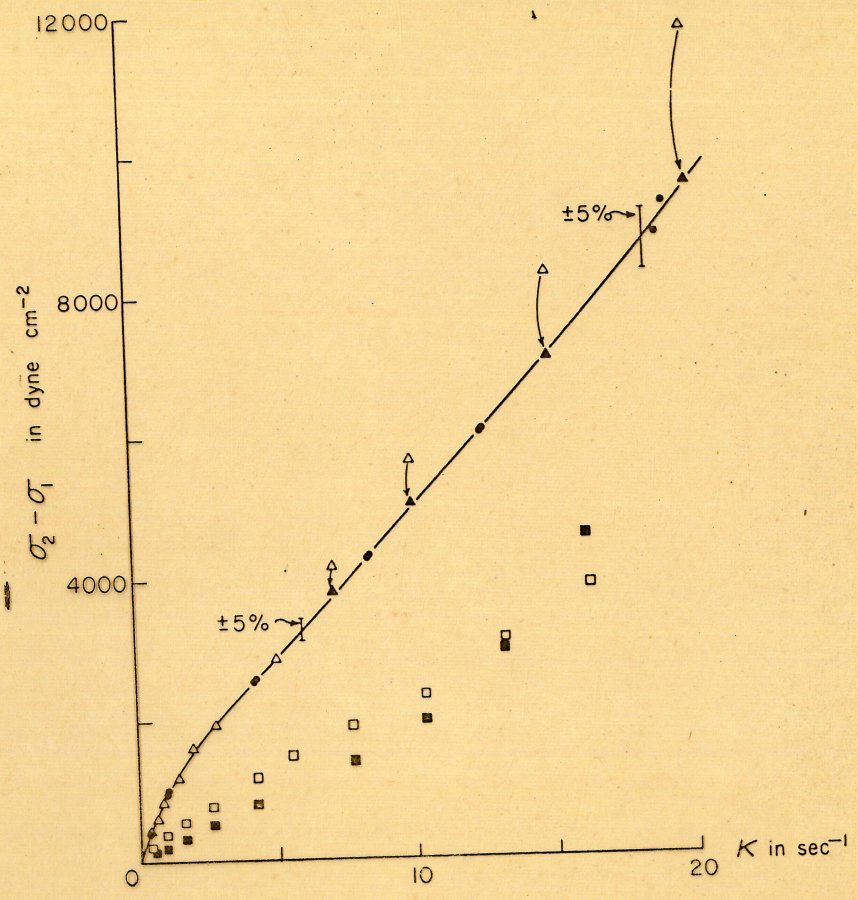

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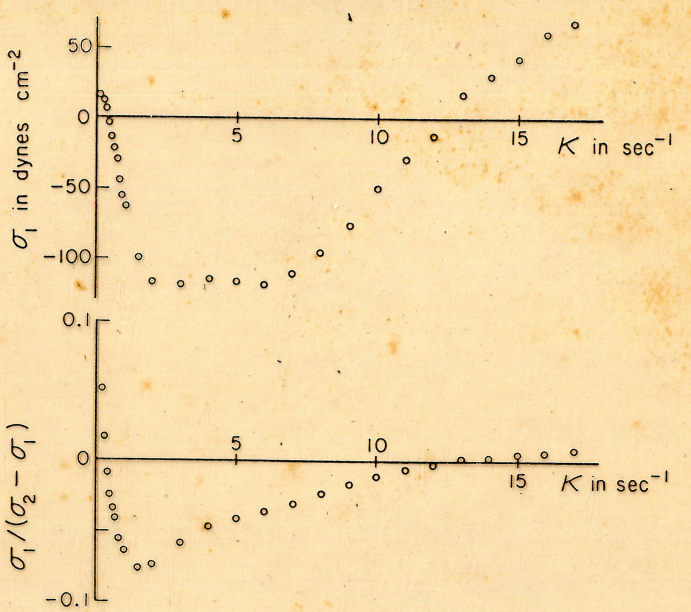

Figure 8 (Courtesy of Professor Pritchard) σ_1 and $\sigma_1/(\sigma_2-\sigma_1)$ as functions of κ, according to Professor Pritchard.

(15 d)

Torsional flow with stationary upper plate
Small holes in the plate into which
manometric tubes are inserted.
Elastical fluid (right side). Levels to which
the fluid rises in the various tubes are
about the same.
Polymer solution (left side). Levels are
greater at the centre than near the outside
edge. Normal pressures exerted by the
upper plate on the fluid to maintain the
plane surface: nearly uniform for N.S.
not for polymer

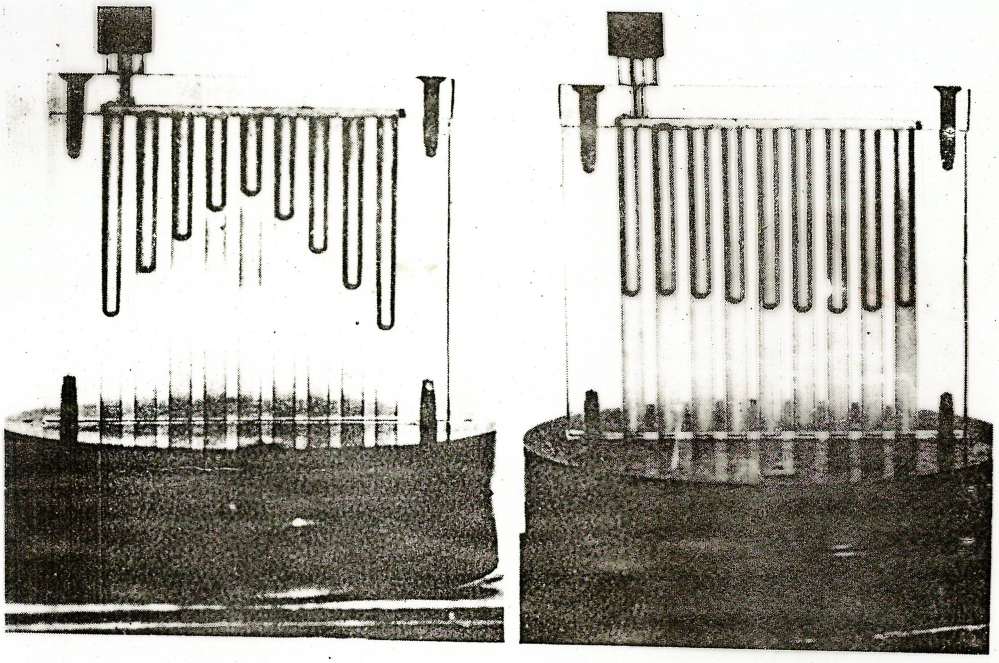

Figure 5 (Courtesy of Professor Lodge and Academic Press) Standpipes in the plate above a fluid in torsional flow. The fluids being tested are those shown in Figure 2.

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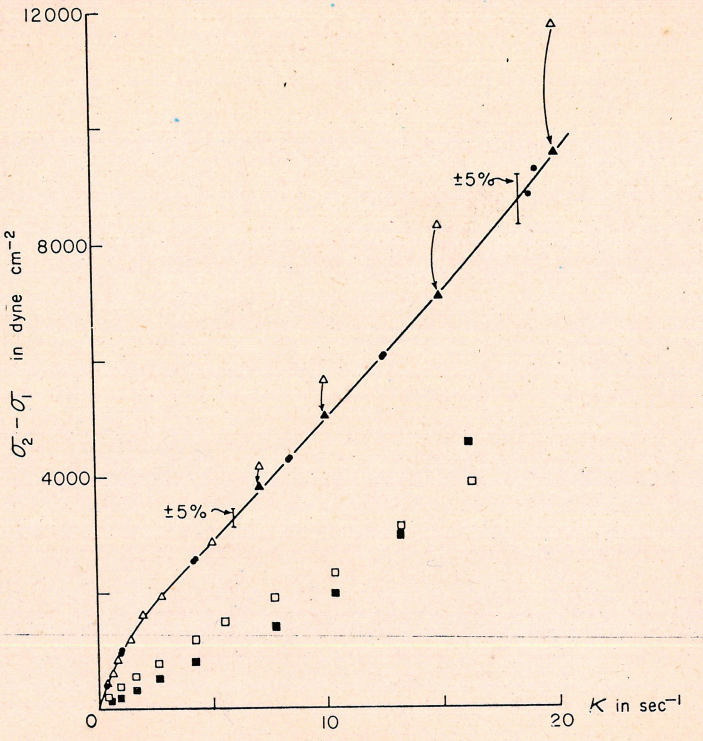

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